APPROXIMATE CALCULATION OF THE CURRENT DISTRIBUTION WHEN A CONDUCTING FLUID FLOWS ALONG A CHANNEL WITHIN A MAGNETIC FIELD

(PRIBLIZHENNYI RASCHET RASPREDELENIIA TOKA PRI TECHENII PROVODIASHCHEI ZHIDKOSTI PO KANALU V MAGNITNOM POLE)

PMM Vol.26, No.3, 1962 pp.548-556

A.B. VATAZHIN and S.A. REGIRER
(Moscow)

(Received March 5, 1962)

This paper considers several problems connected with calculating the three-dimensional distribution of electric current in a conducting medium, moving along a channel in the presence of a magnetic field. It has been found essential to deal with the problem as a three-dimensional one because it is impossible to study within the framework of the one-dimensional theory such problems as the entry of an electroconducting medium into a magnetic field and the effect of non-uniform boundary conditions around the perimeter of a cross section and in the longitudinal direction of the channel. Three-dimensional problems should likewise be dealt with when studying the Hall effect.

It is not practically possible at the present time to arrive at accurate solutions of three-dimensional problems on the basis of the Hall system of magnetohydrodynamic equations, so that to ease the analysis various simplifying models or analogies are created. Approximate solutions [1-7] are constructed only for a few of the simpler problems. In this paper we deal with the problem of the distribution of current when an electroconducting medium flows within channels in the general situation, and then certain assumptions are indicated which lead to simplified systems of solution.

1. Three-dimensional boundary value problems are considerably eased if the hydrodynamic quantities are known.

Thus motions are known in which the hydrodynamic and the electrodynamic equations are separate and they can be solved consecutively. In that case the solution of the electrodynamic equations taking into account the velocity distributions are solved from the hydrodynamic equations and turn out to be exact solutions of the whole system of magnetohydrodynamic equations (Section 2).

If the flow in the channel takes place under conditions of weak magnetohydrodynamic interaction, the hydrodynamic quantities may be considered approximately known from the corresponding solutions of conventional hydrodynamics when there is no magnetic field, and the electromagnetic quantities may be determined with their help.

The same order of solving can also be applied to the case when electromagnetic forces hardly differ from potential ones and therefore to a large degree are evened out. In this case the velocity distribution will hardly differ from that of normal hydrodynamics.

Finally in some cases where there is an arbitrary magnetohydrodynamic interaction there exist approximate expressions for velocity and temperature in the stream; these are obtained either theoretically or from experiment, and they are sufficient for determining the electrical quantities from the Maxwell equations and Ohm's law.

We are going to assume below that one of the above cases prevails and the hydrodynamic quantities are known. Then the steady state problem of current distribution can be written down

$$f(\mathbf{j}, \sigma, \nabla\varphi, \mathbf{B}, \mathbf{v}, \ldots) = 0 \quad (1.1)$$

$$\operatorname{rot} \mathbf{B} = \frac{4\pi\mu}{c}\mathbf{j}, \qquad \operatorname{div} \mathbf{B} = 0 \quad (1.2)$$

From Equations (1.2) the conditions of continuity of electric current density ensues

$$\operatorname{div} \mathbf{j} = 0 \quad (1.3)$$

In this expression φ is the electro-static potential, $\mathbf{B}$ is the electromagnetic induction vector, μ magnetic permeability of the medium (μ = const). Equation (1.1) is in fact a formal expression of Ohm's law. In the general case the conductivity σ may depend not only on $\mathbf{j}$, $\mathbf{B}$, $\nabla\varphi$, the velocity $\mathbf{v}$, the scalar electroconductivity $|\sigma|$, but also on parameters which represent other properties and conditions of the medium. It is evident that all the arguments f, in addition to $\mathbf{B}$, $\mathbf{j}$ and $\nabla\varphi$ are assumed to be known. The system (1.1) to (1.3) serves to determine $\mathbf{B}$, $\mathbf{j}$ and φ. If necessary, after solving this, it is possible to find corrections to the hydrodynamic parameters, solving the hydrodynamic equations with a known body force and heat generation per unit volume.

It should be emphasized that in this paper, as distinct from the so-called "kinematic" problems [8], in which exact solutions of the system (1.1) to (1.3) are sought, we look for approximate solutions on a basis of supplementary assumptions on the properties of the fluid, on the geometry of the current and the character of the magnetic field.

2. We will now consider one class of motions where the velocity distribution of the medium can be accurately determined independently of the equations of electrodynamics. Suppose an incompressible nonviscous fluid moves in an infinitely long plane channel $-\infty < x < +\infty$, $-\delta_1(x) \leqslant y \leqslant \delta_2(x)$ in the presence of an external field $\mathbf{B} = (0,\ 0,\ -B)$. The vector of the external field should satisfy Equations (1.2) when $\mathbf{j}$ is identically equal to 0. (It is assumed that currents which create the field lie outside the region in which the medium is flowing), and it follows from this that $B = B_0 = \text{const}$.

The magnetic field in the fluid may have a z component which need not be constant but depends on x and y. It is known [9] that when there is a magnetic field which is perpendicular to the plane of flow, the electromagnetic force is potential and $\mathbf{j} \times \mathbf{B}/c = -(1/8\pi)\nabla B^2$. In this case the velocity distribution can be found from the well-known equations

$$\operatorname{rot}(\mathbf{v}\times\operatorname{rot}\mathbf{v}) = 0, \qquad \operatorname{div}\mathbf{v} = 0 \tag{2.1}$$

which do not contain electromagnetic terms.

The distribution of current and magnetic field is now found from Equation (1.2) and from Ohm's law (1.1) which, taking account of the Hall effect, can be written in the following form in many cases of practical interest

$$\mathbf{j} = \sigma\left(-\nabla\varphi + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right) - \alpha\mathbf{j}\times\mathbf{B} \qquad \left(\alpha = \frac{\omega\tau}{B}\right) \tag{2.2}$$

Note that if the Hall effect is not taken into account and electrical conductivity is assumed to be infinite, the quantity B_z is easily found with the help of the "frozen flow" integral of [9].

The solution of systems (1.2), (2.2) is considerably simplified also for the case of small magnetic Reynolds numbers $R_m = VL/\nu_m$ (V, L are characteristic velocity and channel dimensions, $\nu_m = c^2/4\pi\mu\sigma$), when the magnetic field in the fluid hardly differs from the external field $\mathbf{B} = (0,\ 0,\ -B_0)$. Then if for simplicity we put $\sigma = \text{const}$, $\alpha = \text{const}$, (and then also $\omega\tau = \text{const}$) and if we take the divergence of (2.2) we arrive at the Poisson equation for the potential

$$\Delta\varphi = \frac{B_0}{c}\left(\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x}\right) \tag{2.3}$$

where $v_x(x, y)$, $v_y(x, y)$ are known from the solution of system (2.1).

The vector components of the current density are expressed through φ thus

$$j_x = \frac{\sigma}{1+\omega^2\tau^2}\left[-\frac{\partial\varphi}{\partial x} + \omega\tau\left(\frac{B_0 v_x}{c} - \frac{\partial\varphi}{\partial y}\right) - \frac{B_0 v_y}{c}\right]$$
$$j_y = \frac{\sigma}{1+\omega^2\tau^2}\left[-\frac{\partial\varphi}{\partial y} + \omega\tau\left(\frac{\partial\varphi}{\partial x} + \frac{B_0 v_y}{c}\right) + \frac{B_0 v_x}{c}\right] \qquad (2.4)$$

When σ and α are given as functions of the coordinates, instead of (2.3) we will arrive at a rather more complicated equation.

The boundary conditions at the walls of the channel consisting of a nonconducting and of ideally conducting sections (dielectrics and electrodes), are formulated in the following way. On the dielectrics there is no current in the direction perpendicular to the walls. Therefore, if we denote the angle between the axis x and the tangent to the wall by ϑ we obtain from (2.4)

$$\frac{\partial\varphi}{\partial x}(\sin\vartheta + \omega\tau\cos\vartheta) - \frac{\partial\varphi}{\partial y}(\cos\vartheta - \omega\tau\sin\vartheta) =$$
$$= \frac{B_0 v_x}{c}(\omega\tau\sin\vartheta - \cos\vartheta) - \frac{B_0 v_y}{c}(\sin\vartheta + \omega\tau\cos\vartheta) \qquad (2.5)$$

(on the dielectrics)

At the electrodes, because of the assumption of ideal conductivity, φ = const. The values of these constants are tied up with parameters of the external electric circuit through Ohm's law (see Section 3).

If we assumed that the channels were of infinite length, it is essential further to lay down asymptotic conditions, i.e. conditions at infinity above and below the stream. Suppose for instance that at infinity the walls of the channel are electrodes, the distance between which is constant, and the Hall currents flow freely. Then $E_x = -\partial\varphi/\partial x = 0$ for $|x| \to \infty$. If at infinity the walls are dielectrics and the conditions are such that a division takes place between the charges, then we get $j_x = j_y = 0$ when $|x| \to \infty$.

For the case where the channel has constant width and at infinity $v_y = 0$, $v_x = V$ = const, this velocity distribution, as follows from Equation (2.1), is maintained along the whole channel. Introducing the function $\varphi_1 = \varphi - BVy/c$, we get

$$\Delta\varphi_1 = 0, \qquad \frac{\partial\varphi_1}{\partial x} = 0 \quad \text{(at the electrodes)} \qquad (2.6)$$
$$\omega\tau\frac{\partial\varphi_1}{\partial x} = \frac{\partial\varphi_1}{\partial y} \quad \text{(on the dielectrics)}$$

System (2.6) was deduced and solved in [2] for two special problems.

Note moreover that a similar investigation can be carried out for the flow of a viscous fluid; in this case instead of (2.1) we will have the Helmholtz equation.

3. Let us now take a look at fluid flows which take place in the presence of a three-dimensional magnetic field when the considerations of the preceding section no longer hold.

We deal with the system (1.1) to (1.3) for an isotropic conductive fluid, when Ohm's law can be written down thus

$$\mathbf{j} = \sigma\left(-\nabla\varphi + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right) \qquad (3.1)$$

where σ and $\mathbf{v}$, in agreement with the initial basic assumption, are given functions of the coordinates. If we apply to (3.1) the operation div we arrive at the equation

$$\nabla\sigma\left(-\nabla\varphi + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right) - \sigma\Delta\varphi + \frac{\sigma}{c}(\mathbf{B}\operatorname{rot}\mathbf{v} - \mathbf{v}\operatorname{rot}\mathbf{B}) = 0$$

which, using (1.2), can be brought to the following form

$$\Delta\varphi = \nabla\ln\sigma\left(-\nabla\varphi + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right) + \frac{1}{c}\mathbf{B}\operatorname{rot}\mathbf{v} + \frac{1}{\nu_m}\mathbf{v}\nabla\varphi \qquad (3.2)$$

For low magnetic Reynolds numbers $R_m = VL/\nu_m \ll 1$ the last term in (3.2) can be neglected, then

$$\Delta\varphi = \nabla\ln\sigma\left(-\nabla\varphi + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right) + \frac{\mathbf{B}}{c}\operatorname{rot}\mathbf{v} \qquad (3.3)$$

Here the magnetic field strength $\mathbf{B}$ can be considered known and approximately equals the magnetic field which is applied externally. With given values of $\mathbf{v}$ and σ Equation (3.3), with corresponding boundary conditions allows the potential φ to be found, and with the assistance of (3.1), the current $\mathbf{j}$.

The solution of systems (3.1), (1.2), (1.3) becomes much more difficult if $R_m \gtrsim 1$. Because in such a case the external field is distorted by the motion of the medium and the quantity $\mathbf{B}$ is unknown Equation (3.2) should be solved together with Equations (3.1) and (1.2). Sometimes however the problem of determining the potential can be separated from the determination of the magnetic field. For instance if σ = const, rot $\mathbf{v} = 0$ or $(\mathbf{v}\times\mathbf{B})\nabla\log\sigma = 0$, rot $\mathbf{v} = 0$ whilst only the components of the external magnetic field enter the boundary conditions, then the equations

$$\Delta\varphi = \frac{1}{\nu_m}\mathbf{v}\nabla\varphi \quad \text{when} \quad \Delta\varphi = -\nabla\ln\sigma\nabla\varphi + \frac{1}{\nu_m}\mathbf{v}\nabla\varphi \qquad (3.4)$$

can be solved independently of the others. Furthermore, from the Equations (3.5)

$$\nu_m \operatorname{rot} \mathbf{B} = -c \nabla \varphi + \mathbf{v} \times \mathbf{B}, \qquad \operatorname{div} \mathbf{B} = 0 \tag{3.5}$$

the magnetic field strength $\mathbf{B}$ is found, and, from (3.1) the current $\mathbf{j}$. One such problem is solved in [3].

In cases where it is not possible to separate the equations in this manner, it is sometimes still permissible to make one further assumption, namely to assume that the distribution of the magnetic field is known, and thus, again, only to deal with Equations (3.1), (3.2). This assertion becomes clearer if we consider the approximate method of solution as the first stage in some system of successive approximations. The accuracy of the results then will depend on the accuracy of the quantities $\mathbf{v}$, σ and $\mathbf{B}$ which are fed into the calculation, whilst the error can be assessed by inspection of the equation of the next approximation.

It should be noted that the magnetic field derivatives have been excluded from Equation (3.2). If these were retained the accuracy of the approximate solution might suffer considerably, for, with a small error given in the field, the error in the given derivatives might be large.

Going over now to the formulation of the boundary conditions we assume that the channel walls consist of sections with differing, finite, conductivity. At the boundary between fluid and wall continuity conditions should be fulfilled for the normal component of current density and the tangential component of the electric field $\mathbf{E} = -\nabla\varphi$

$$\sigma\left[-\frac{\partial \varphi}{\partial n} + \frac{1}{c}(\mathbf{v} \times \mathbf{B})_n\right] = -\sigma^* \frac{\partial \varphi^*}{\partial n} \tag{3.6}$$

$$\boldsymbol{\tau}_1 \frac{\partial \varphi}{\partial \tau_1} + \boldsymbol{\tau}_2 \frac{\partial \varphi}{\partial \tau_2} = \boldsymbol{\tau}_1 \frac{\partial \varphi^*}{\partial \tau_1} + \boldsymbol{\tau}_2 \frac{\partial \varphi^*}{\partial \tau_2} \tag{3.7}$$

Here φ^* is the potential at the channel wall, σ^* its conductivity τ_1 and τ_2 orthogonal unit vectors in the plane tangential to the surfaces separating the media. The distribution of potential at the wall is consistent with an equation of the type (3.2)

$$\sigma^* \triangle \varphi^* = -\nabla \sigma^* \nabla \varphi^* \tag{3.8}$$

and also fulfils additional conditions at the external boundary of the wall. It is easy to see that when $\sigma^* \to 0$ and $\sigma^* \to \infty$ we arrive at the well-known conditions

$$j_n = 0, \qquad \frac{\partial \varphi}{\partial n} = \frac{1}{c}(\mathbf{v} \times \mathbf{B})_n \qquad \text{(on the dielectrics)} \tag{3.9}$$

$$\varphi = \text{const} \quad \text{(on the electrodes)} \tag{3.10}$$

If two electrodes are connected to each other through an external circuit with resistance R, and a current J flows from the electrode at potential φ_1 to the electrode at potential φ_2, then the quantities $\varphi_1 - \varphi_2$, J and R are connected by Ohm's law

$$\varphi_1 - \varphi_2 = JR, \qquad J = \int_{S_1} j_n dS = \int_{S_2} j_n dS \tag{3.11}$$

where S_i are electrode areas.

Asymptotic conditions for φ are obtained from (3.1). Suppose, for instance, the principal current direction coincides with the x-axis, whilst the magnetic field vanishes at infinity. Then both current and potential will also tend to zero

$$\varphi = 0, \quad \nabla \varphi = 0 \text{ when } x = \pm\infty \tag{3.12}$$

If charge separation takes place at infinity, and, therefore the current vanishes, we have

$$\nabla\varphi = \frac{1}{c}\, \mathbf{v} \times \mathbf{B} \text{ when } x = \pm\infty \tag{3.13}$$

Any further simplification of the equations written down is usually associated with assumptions about straight fluid streamlines: $\mathbf{v} = (v, 0, 0)$. The magnetic field strength component, parallel to the velocity, will not then enter the calculation and it becomes easier to specify $\mathbf{B}$. However the solution of three-dimensional problems remains difficult with Equations (3.2) and (3.3). For this reason we deal with two-dimensional problems in which the effects of longitudinal and of transverse flows are treated separately.

4. It is especially easy to go over to two-dimensional problems when the conductivity is constant and for small values of R_m. Assume that a fluid flows along a rectangular channel of infinite length, the section being $|y| < \delta(x)$, $|z| < a(x)$, whilst the velocity profile, the magnetic field and the boundary conditions are symmetrical with respect to the plane $z = 0$, so that the potential φ is, at the outset, an even function of z. Then if we define the operation of taking the mean value with respect to z in this manner

$$\langle w \rangle = \frac{1}{2a} \int_{-a}^{+a} w\,(x, y, z)\, dz$$

and apply it as if it were (3.1) and (3.3) we get

$$\langle \mathbf{j} \rangle = \sigma\left(-\nabla \langle \varphi \rangle + \frac{1}{c} \langle \mathbf{v} \times \mathbf{B} \rangle\right) \tag{4.1}$$

$$\triangle \langle \varphi \rangle = \frac{1}{c} \langle \mathbf{B} \operatorname{rot} \mathbf{v} \rangle - \frac{1}{a} \left| \frac{\partial \varphi}{\partial z} \right|_{z=a} \tag{4.2}$$

These two equations together with the boundary conditions at the walls $y = \pm \delta$ taken as averaged with respect to z, allow indeed of a solution to the two-dimensional longitudinal problem at infinity.

In some cases it is permissible to neglect correlations when working out mean values of a product, i.e. to take approximately

$$\langle w_1 w_2 \rangle = \langle w_1 \rangle \langle w_2 \rangle$$

Then Equations (4.1) and (4.2) will take the form of Equations (3.1) and (3.3) if in the latter we formally put $j_z = 0$, $\partial/\partial z = 0$, $\sigma = \text{const}$, $\mathbf{j} = \langle \mathbf{j} \rangle$, $\varphi = \langle \varphi \rangle$ and add the component

$$\frac{1}{a} \left| \frac{\partial \varphi}{\partial z} \right|_{z=a} = -\frac{1}{a} \left(\frac{j_z}{\sigma} - \frac{1}{c} v B_y \right)_{z=a} \tag{4.3}$$

This additional term vanishes exactly, for instance, when the walls $z = \pm a$ are nonconducting and on them $v = 0$ or $B_y = 0$. In the general case Equation (4.3) together with the boundary conditions for $z = \pm a$ allow of the exclusion of the unknown $\partial\varphi/\partial z$ from (4.2).

A similar procedure involving taking the mean with respect to coordinate x over some interval (along the length of the electrode, etc.) leads to two-dimensional transverse problems. When we have variable conductivity, idependent of the coordinates along which the mean is taken, all the arguments still remain valid. In the worst case two-dimensional problems of the type (4.1), (4.2) can only be resolved by neglecting correlations when taking the mean of products containing σ.

Let us take a closer look at problems with straight line fluid flow at small R_m in a constant section rectangular channel. The channel walls $|z| = a$ are everywhere nonconducting, whilst at the walls $|y| = \delta$ there are symmetrically positioned electrodes. Suppose a magnetic field is created by a magnet whose poles are bounded by planes $z = \pm z_1$, $|x| < x_1$, $|y| < y_1$, for which $y_1 < \delta$. To a considerable degree of accuracy, then, it is possible to say that within the region of flow $B_x = B_x(x, z)$, $B_y = 0$, $B_z = B_z(x, z)$ and B_z is an even function of z. When flow takes place in the operation of electric current generation B_z is the "working" component of the magnetic field.

The assumptions which have been made and the condition of rectilinear flow allow Equations (3.1) and (3.3) to be written down in a form containing only the working component of the field

$$j_x = -\sigma \frac{\partial \varphi}{\partial x}, \qquad j_y = \sigma\left(-\frac{\partial \varphi}{\partial y} - \frac{1}{c} vB_z\right), \qquad j_z = -\sigma \frac{\partial \varphi}{\partial z} \tag{4.4}$$

$$\Delta\varphi = -\nabla \ln \sigma \nabla \varphi - \frac{1}{c}\frac{\partial \ln \sigma}{\partial y} vB_z - \frac{B_z}{c}\frac{\partial v}{\partial y} \tag{4.5}$$

With constant conductivity and velocity of flow, by taking the mean in z, bearing in mind (3.9), we get

$$\langle j_x\rangle = -\sigma \frac{\partial}{\partial x}\langle\varphi\rangle, \qquad \langle j_y\rangle = \sigma\left(-\frac{\partial}{\partial x}\langle\varphi\rangle - \left\langle\frac{vB_z}{c}\right\rangle\right) \tag{4.6}$$
$$\Delta\langle\varphi\rangle = 0$$

The boundary conditions for $y = \pm\delta$ change their form just as integral condition (3.11); in the latter integrals x is taken within the limits of the electrode.

The system (4.6), which represents the electric field and the current distribution "averaged out" over the width of the channel, is similar in form to the system of equations used in [4-6] for solving plane problems. Thus the results of these papers are not only applicable to plane problems but to three-dimensional ones as well.

5. We now deal with flow of a fluid with anisotropic conductivity when Ohm's law has the form (2.1)

$$\mathbf{j} = \sigma\left(-\nabla\varphi + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right) - \alpha\mathbf{j}\times\mathbf{B} \tag{5.1}$$

The magnetic field here, as distinct from Section 2, is assumed to be three-dimensional. From (5.1) an explicit expression for $\mathbf{j}$ can be found

$$\mathbf{j} = \frac{\sigma}{1+\alpha^2B^2}\left[\mathbf{E}' + \alpha\mathbf{B}\times\mathbf{E}' + \alpha^2\mathbf{B}(\mathbf{E}'\mathbf{B})\right]$$

where $\mathbf{E}' = -\nabla\varphi + \mathbf{v}\times\mathbf{B}/c$, but further transformation to the equation for φ turns out to be very difficult and, in general, does not lead to acceptable results. In particular it is not possible to eliminate from the equation the magnetic field derivatives. We therefore first of all transform Expression (5.1) with the help of the equations of motion

$$\nabla p = \mathbf{F} + \frac{1}{c}\mathbf{j}\times\mathbf{B} \tag{5.2}$$

where $\mathbf{F}$ is the sum of the inertia and viscous forces. If we eliminate $\mathbf{j}\times\mathbf{B}$ from (5.1) and (5.2) we will obtain

$$\mathbf{j} = \sigma\left(-\nabla\Phi + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right) + \alpha c\mathbf{F} \qquad \left(\Phi = \varphi + \frac{\alpha c}{\sigma}p\right) \tag{5.3}$$

Now we apply operation div to (5.3), and assuming the magnetic Reynolds number to be small, we arrive at an equation similar to (3.3)

$$\triangle\Phi = \frac{1}{c}\mathbf{B}\ \mathrm{rot}\ \mathbf{v} + \alpha c\ \mathrm{div}\,\mathbf{F} \tag{5.4}$$

The vector **F** can here be assumed given because it is expressed through known hydrodynamic quantities (velocity **v** and its derivatives).

The boundary conditions for Φ on the dielectric are easy to find if we insist that the normal current component vanishes

$$\frac{\partial\Phi}{\partial n} = \frac{\alpha c}{\sigma}F_n + \frac{1}{c}(\mathbf{v}\times\mathbf{B})_n \quad \text{(on the dielectric)} \tag{5.5}$$

To obtain the conditions on the electrode we eliminate $\mathbf{j}$ from (5.2) and (5.3) and we project the equations obtained on the tangents in directions τ_1, τ_2 to the electrode surface. We then observe that on this surface

$$\varphi = \mathrm{const}, \qquad \frac{\partial\Phi}{\partial\tau_i} = \frac{\alpha c}{\sigma}\frac{\partial p}{\partial\tau_i} \qquad (i = 1,2) \tag{5.6}$$

and we get

$$\frac{\sigma}{\alpha}\frac{\partial\Phi}{\partial\tau_i} + cF_{\tau_i} = \left|\left[\sigma\left(-\nabla\Phi + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right) + \alpha c\mathbf{F}\right]\times\mathbf{B}\right|_{\tau_i} \qquad (i = 1,2) \tag{5.7}$$

(on the electrodes)

It follows from (5.7), in particular, that here the field parallel to the velocity component remains, in general, in the equations of the problem.

The condition connecting the potential difference of the two electrodes with the external circuit parameters, and expressed through Φ can be obtained from the following reasoning. Suppose, for instance, that the electrodes be located symmetrically at the walls $y = \pm\,\delta$ of a rectangular channel Then by definition

$$\varphi(\delta) - \varphi(-\delta) = \Phi(x, \delta, z) - \Phi(x, -\delta, z) - \frac{\alpha c}{\sigma}\left[p(x, \delta, z) - p(x, -\delta, z)\right]$$

The second factor on the R.H.S. can be transformed with the assistance of (5.2) whilst the L.H.S. is replaced by JR. Finally we arrive at

$$JR = \Phi(x, \delta, z) - \Phi(x, -\delta, z) - \frac{\alpha c}{\sigma}\int_{-\delta}^{\delta} \left[F_y + \frac{1}{c}(\mathbf{j}\times\mathbf{B})_y\right] dy \quad (5.8)$$

This equation (it might have a different form with channels of a different shape) replaces condition (3.11) for the case of an anisotropic conducting medium.

The asymptotic conditions for the supplementary potential may be obtained easily from (5.3). In particular, assuming no magnetic field or current at infinity

$$\nabla\Phi = \frac{\alpha c}{\sigma}\mathbf{F} \qquad \text{when } x = \pm\infty \quad (5.9)$$

The transition from three-dimensional to two-dimensional problems in the foregoing equations can also be made by the method of taking the mean. For instance let us deal with the special case of the flow in a constant section rectangular channel, when, as in Section 4, $\mathbf{v} = (v,0,0)$, the condition of adherence to the walls is observed and the magnetic field has only components B_x and B_z depending on x and z. After taking the mean, instead of Equation (5.4) we obtain the two-dimensional Poisson equation, and instead of the two conditions (5.7), only one

$$\frac{\partial}{\partial x}\langle\Phi\rangle + \frac{\alpha c}{\sigma}\langle F_x\rangle = -\alpha\langle B_z\rangle\frac{\partial}{\partial y}\langle\Phi\rangle \quad \text{(on the electrodes)} \quad (5.10)$$

Observe that with this magnetic field structure the integral in (5.8) reduces to the differences of the magnetic pressures for δ and $-\delta$, which with a symmetrical external field and small induced fields, equals zero:

Thus condition (5.10) is also considerably simplified and is reduced to the form (3.11).

It should be observed that the considerations outlined in this section, can, by analogy with Section 3, be extended to the case where σ and α be given functions of the coordinates. Additionally Ohm's law can be exploited in a more general form which includes the electron pressure p_e

$$\mathbf{j} = \sigma\left(-\nabla\varphi + \frac{1}{c}\mathbf{v}\times\mathbf{B} + \frac{1}{en_e}\nabla p_e\right) - \alpha\mathbf{j}\times\mathbf{B}$$

If we introduce the supplementary potential

$$\Phi = \varphi - \frac{\alpha c}{\sigma}p - \frac{p_e}{en_e}$$

and we assume that $p_e = \zeta p$, ζ = const, then only the constant multipliers change in some of the foregoing equations. For a fully ionized gas the assumption of constant ζ means that the ratio of the electron and ion component temperatures is constant over the whole volume, and $\zeta = T_e/(T_e + T_i)$.

Finally we demonstrate that the problems dealt with in this paper reduce, at low magnetic Reynolds numbers to the Poisson equation or to a nonhomogeneous elliptical equation of a more general type with linear boundary conditions. Additionally, when dealing with actual problems it is better to make use of homogeneous equations for which there exist effective methods of solution based on the theory of the complex variable. It would be interesting therefore to study problems on the flow in channels with dielectric walls, as then the simpler special solutions of nonhomogeneous equations can be obtained, which are essential when going over to the homogeneous equations.

BIBLIOGRAPHY

1. Resler, E. and Sears, U., Perspektivy magnitnoi aerodinamiki (Perspectives in magneto-aerodynamics). *Sb. per Mekhanika* No. 6 (52), pp. 3-22, 1958.

2. Hurwitz, H., Jr., Kilb, R.W. and Sutton, G.W., Influence of tensor conductivity on current distribution in a MHD generator. *J. Appl. Phys.*, Vol. 32, No. 2, pp. 205-216, 1961.

3. Boucher, R.A. and Ames, D.B., End effect losses in d.c. magnetohydrodynamic generators. *J. Appl. Phys.* Vol. 32, No. 5, pp. 755-759, 1961.

4. Birzvalk, Iu.A., Ekvivalentnaia skhema kanala nasosa postoiannogo toka i raschet nasosa na maksimum k.p.d. (Equivalent circuit of a d.c. pump and calculation for maximum efficiency of pump). *Tr. In-ta fiziki Akada Nauk Latv SSR* Vol. 12, pp. 25-141, 1961.

5. Vatazhin, A.B., K resheniiu nekotorykh kraevykh zadach magnitogidrodinamiki (Solution of several boundary problems in magnetohydrodynamics). *PMM* Vol. 25, No. 5, 1961.

6. Vatazhin, A.B., Magnitogidrodinamicheskoe techenie v ploskom kanale s konechnymi elektrodami (Magnetohydrodynamic flow in a plane channel with finite electrodes). *Izv. Akad. Nauk SSSR OTN Mekhanika i Mashinostroenie* No. 1, pp. 52-58, 1962.

7. Sutton, G.W. and Carlson, A.W., End effects in inviscid flow in a magnetohydrodynamic channel. *J. Fluid Mech.* Vol. 11, No. 1, pp. 121-132, 1961.

8. Cowley, M.D., On some kinematic problems in magnetohydrodynamics. *Quart. J. Mech. and Appl. Math.* Vol. 14, No. 3, pp. 319-333, 1961.

9. Grad, H., Reducible problems in magnetofluid dynamic steady flows. *Rev. Mod. Phys.* Vol. 32, No. 4, pp. 830-847, 1960.

10. Liubimov, G.A., O forme zakona Oma v magnitnoi gidrodinamike (On the form of Ohm's law in magnetohydrodynamics). *PMM* Vol. 25, No. 4, 1961.

Translated by V.H.B.